Classification of integrable Volterra type lattices on the sphere. Isotropic case

V.E. Adler*

17 December 2007

Abstract

The symmetry approach is used for classification of integrable isotropic vector Volterra lattices on the sphere. The list of integrable lattices consists mainly of new equations. Their symplectic structure and associated PDE of vector NLS-type are discussed.

1 Introduction

We call vector Volterra lattices the equations of the general form

$$V_{n,x} = f_n V_{n+1} + g_n V_n + h_n V_{n-1}, \quad n \in \mathbb{Z} \tag{1}$$

where V_n are vectors and f_n, g_n, h_n are scalar functions depending on V_{n+1}, V_n, V_{n-1}. The integrability is understood as existence of higher symmetries, that is the equations which are consistent with (1), but involve the larger number of neighbor vectors (preserving the same quasi-linear structure). The precise definitions are given in next Section. The goal of this paper is to classify integrable cases under the following assumptions:

(i) the lattice and its symmetries are isotropic and shift invariant, that is their coefficients depend only on the scalar products $v_{m,n} := \langle V_m, V_n\rangle = \langle V_n, V_m\rangle$ and this dependence is the same at each node;

(ii) the lattice must be integrable independently on the dimension of the vector space and the nature of scalar product;

(iii) all V_n are of unit length, $v_{n,n} = 1$.

The shift invariance allows the use of shorthand notation with the discrete variable n omitted from subscripts, so that equation (1) takes the form

$$V_x = fV_1 + gV + hV_{-1} \tag{2}$$

(subscripts x, t will be reserved for denoting derivatives, not shifts). Due to the other assumptions, functions f, g, h are related by equation

$$v_{1,0}f + g + v_{0,-1}h = 0 \tag{3}$$

*L.D. Landau Institute for Theoretical Physics, 1a Semenov pr., 142432 Chernogolovka, Russia. E-mail: adler@itp.ac.ru

and depend only on the scalar products $v_{1,0}, v_{0,-1}, v_{1,-1}$ which can be considered as independent variables. Therefore, the classification problem is reduced to finding of two functions of three variables, so that its complexity is comparable with the case of scalar Volterra lattices

$$v_x = f(v_1, v, v_{-1}) \tag{4}$$

classified by Yamilov [1], see also the recent review article [2]. The whole method of solution is also very close, since the necessary integrability conditions in both cases formally coincide (the difference is in the set of dynamical variables: $v_{m,n}$ instead of v_n). In the continuous case, the general approach based on this remarkable observation has been developed by Sokolov and Meshkov in the pioneering papers [3, 4] devoted to the classification of KdV-type vector equations (including the anisotropic ones) on the sphere. Important classification results for some other types of vector PDE were obtained in [5, 6, 7], however the approach in these papers relied essentially on the polynomial or rational structure of equations.

In principle, the classification problem for the lattices (2) can be solved without the unitary condition (iii). This constraint does not define an independent class of equations, but only a special reduction of the general problem. Indeed, it can be resolved by use of the stereographic projection

$$V = \frac{1 - \langle U, U\rangle}{1 + \langle U, U\rangle}\, e_0 + \frac{2}{1 + \langle U, U\rangle} U,$$

where e_0 is some fixed unit vector and U belongs to its orthogonal subspace. Vector U satisfies, in virtue of equation (2), some isotropic lattice $U_x = \tilde{f}U_1 + \tilde{g}U + \tilde{h}U_{-1}$. Since the dimension of the vector space is inessential in our considerations, we see that any lattice on the sphere corresponds under this mapping to some lattice in the free space. On the other hand, this lattice for U is not arbitrary: it must admit the reduction $\langle U, U\rangle = 1$ since $U = V$ under this constraint. This reduction brings back to the original lattice.

The paper is organized as follows. Section 2 contains a concise explanation of symmetry approach and derivation of the sequence of integrability conditions in the form of conservation laws. These are used in Section 3 which is the main technical body of the text. All lattices (2) are divided there into two subclasses; the first one is analyzed thoroughly, while the second one is poor in answers and its presentation is more brief. The results of classification are presented in Section 4. The rest of the paper contains some discussion of associated PDEs and symplectic structures.

2 The necessary integrability conditions

The symmetry approach to classification of integrable equations had been developed in 80's, see e.g. [9, 10] as general sources, [13] for a modern account on the discrete case and review articles [14, 2] for detailed references. The lattice (2) is called integrable if it possesses an infinite hierarchy of the symmetries of the form

$$V_{t_k} = p^{(k,k)}V_k + p^{(k,k-1)}V_{k-1} + \cdots + p^{(k,1-k)}V_{1-k} + p^{(k,-k)}V_{-k} \tag{5}$$

with coefficients depending on the scalar products of $V_k, \dots, V_{-k}$. It is easy to see that the compatibility condition splits over the vector variables V_n resulting in the commutator

relation

$$D_x(P^{(k)}) - D_{t_k}(F) = [F, P] \tag{6}$$

for *scalar* operators

$$F = fT + g + hT^{-1}, \quad P^{(k)} = p^{(k,k)}T^k + \dots + p^{(k,-k)}T^{-k}$$

where T denotes the shift operator $n \mapsto n+1$. This allows to use the necessary integrability conditions established in the scalar case (4) by Yamilov [1], with operator F instead of the linearization operator $f_* = f_{v_1}T + f_v + f_{v_{-1}}T^{-1}$. For sake of completeness we repeat very briefly the derivation of these conditions.

Equation (6) is equivalent to a set of equations for the coefficients of $P^{(k)}$. One pair of equations defines explicitly the leading coefficients

$$p^{(k,k)} = f_{k-1}\dots f_1 f, \quad p^{(k,-k)} = \alpha h_{-k+1}\dots h_{-1}h,$$

while solvability of the rest equations provides some sequence of necessary conditions to integrability of the lattice. These conditions do not depend actually on the order k of the symmetry. More precisely, let equation (6) can be solved, at some $k = K$, with respect to $2l$ coefficients $p^{(k,\pm k)}, p^{(k,\pm(k-1))}, \dots, p^{(k,\pm(k-l))}$, where $k - l > 1$. Then it can be solved with respect to these $2l$ coefficients at any $k > K$. Moreover, the coefficients of one symmetry are expressed through the coefficients of the other one by explicit formulae. In order to prove this, it is sufficient to notice that the term $D_{t_k}(F)$ in the l.h.s. of (6) affects the computation of the coefficients $p^{(k,1)}, p^{(k,0)}, p^{(k,-1)}$ only, and that the special form of the leading coefficients written above allows to approximate $P^{(k)}$ by the formal power series $(P^{(K)})^{k/K}$. This brings to the following statement.

Statement 1. *If the lattice (2) possesses an infinite hierarchy of higher symmetries then the equations*

$$\begin{aligned} L_x &= [F, L], \quad L = a^{(-1)}T + a^{(0)} + a^{(1)}T^{-1} + a^{(2)}T^{-2}\dots \\ \tilde{L}_x &= [F, \tilde{L}], \quad \tilde{L} = \tilde{a}^{(-1)}T^{-1} + \tilde{a}^{(0)} + \tilde{a}^{(1)}T + \tilde{a}^{(2)}T^2\dots \end{aligned}$$

are solvable with respect to the coefficients $a^{(j)}$, $\tilde{a}^{(j)}$ depending on $v_{m,n}$.

The series $L, \tilde{L}$ are called *formal symmetries*. In turn, the equations for their coefficients can be rewritten further as the sequence of conservation laws

$$D_x(\rho^{(j)}) = (T-1)(\sigma^{(j)}), \quad D_x(\tilde{\rho}^{(j)}) = (T^{-1}-1)(\tilde{\sigma}^{(j)}), \quad j = 0, 1, 2, \dots \tag{7}$$

More precisely, if the lattice (2) possesses the symmetry of order k, then equations (7) can be solved with respect to $\sigma^{(j)}, \tilde{\sigma}^{(j)}$ for $j = 0, \dots, k-2$. The densities $\rho^{(j)}, \tilde{\rho}^{(j)}$ are expressed explicitly by certain recursive algorithm in terms of the lattice coefficients and previously found $\sigma^{(j)}, \tilde{\sigma}^{(j)}$. This algorithm relates $\rho^{(j)}$ with the residue of L^j defined as the free term of power series in T (the formula $\mathrm{res}[A, B] \in \mathrm{Im}(T-1)$ can be proven). However, in practice we will need only few several conservation laws and the corresponding formulae can be derived straightforwardly.

Statement 2. *Let the lattice (2) be integrable, then equations (7) are solvable for the following sequence of the densities $\rho^{(j)}$, $\tilde{\rho}^{(j)}$:*

$$\rho^{(0)} = \log f, \qquad \tilde{\rho}^{(0)} = \log h, \tag{8}$$
$$\rho^{(1)} = g + \sigma^{(0)}, \qquad \tilde{\rho}^{(1)} = g + \tilde{\sigma}^{(0)}, \tag{9}$$
$$\rho^{(2)} = hf_{-1} + \frac{1}{2}(\rho^{(1)})^2 + \sigma^{(1)}, \qquad \tilde{\rho}^{(2)} = fh_1 + \frac{1}{2}(\tilde{\rho}^{(1)})^2 + \tilde{\sigma}^{(1)}. \tag{10}$$

Proof. The equations for the coefficients $a^{(-1)}, a^{(0)}, a^{(1)}, a^{(2)}$ are:

$$\begin{aligned}
0 &= fa_1^{(-1)} - f_1 a^{(-1)},\\
a_x^{(-1)} &= fa_1^{(0)} - fa^{(0)} + ga^{(-1)} - g_1 a^{(-1)},\\
a_x^{(0)} &= fa_1^{(1)} - f_{-1}a^{(1)} + ha_{-1}^{(-1)} - h_1 a^{(-1)},\\
a_x^{(1)} &= fa_1^{(2)} - f_{-2}a^{(2)} + ga^{(1)} - g_{-1}a^{(1)} + ha_{-1}^{(0)} - ha^{(0)}.
\end{aligned}$$

The first equation implies $a^{(-1)} = f$, without loss of generality. Then the second equation takes the form $(\log f)_x = (T-1)(a^{(0)} - g)$, so that we obtain the density $\rho^{(0)}$ and the formula for the next coefficient of the formal symmetry: $a^{(0)} = g + \sigma^{(0)}$. Accordingly to the third equation, this coefficient may be taken as the density $\rho^{(1)}$ and then $a^{(1)} = h + \sigma^{(1)}/f_{-1}$. The last equation can be brought to the form

$$\Big(hf_{-1} + \frac{1}{2}(\rho^{(1)})^2 + \sigma^{(1)}\Big)_x = (T-1)(f_{-1}f_{-2}a^{(2)} + \sigma^{(1)}\rho_{-1}^{(1)})$$

after multiplication by f_{-1} and taking into account the previous equations. The second set of the densities is obtained immediately due to the symmetry $n \to -n$. □

Remark 1. In addition to the higher symmetries, existence of the higher order conservation laws is another characteristic feature of integrable equations. It is possible to derive some integrability conditions from this property as well. This leads to the notion of *formal conservation law*

$$S_x + SF + F^\top S = 0, \quad S = s^{(0)} + s^{(1)}T^{-1} + s^{(2)}T^{-2} + \dots$$

where $(aT^j)^\top := T^{-j}a$ and coefficients $s^{(j)}$ depend on $v_{m,n}$. Solvability of this equation is equivalent to the sequence of conditions of the form

$$\hat{\rho}^{(j)} = (T-1)(\hat{\sigma}^{(j)}), \quad j = 0, 1, 2, \dots \tag{11}$$

In particular,

$$\hat{\rho}^{(0)} = \log(-f/h), \quad \hat{\rho}^{(1)} = 2g + D_x(\hat{\sigma}^{(0)}).$$

It can be proven that conservation laws (7) are equivalent in virtue of conditions (11), that is $\rho^{(j)} + \mathrm{const}\,\tilde{\rho}^{(j)} \in \mathbb{C} \oplus \mathrm{Im}(T-1)$. In some classification problems use of these additional integrability conditions may lead to a crucial simplification or even to a shorter list of equations. In particular, these conditions were used by Yamilov in his classification of the scalar lattices (4) (see footnote on p. 567 and Theorem 22 in [2]). It turns out, however, that in the vector case these conditions are of minimal value and it is possible to dispense with them (in all found lattices they are fulfilled automatically).

Returning to the characteristic equation (6) we notice that solvability of the first pair of integrability conditions (7), (8) allows to find the coefficients $p^{(k,\pm k)}$, $p^{(k,\pm(k-1))}$ of the symmetry. At $k=2$ this defines the symmetry completely, due to the constraint $\langle V,V\rangle=1$ which implies

$$v_{2,0}p^{(2,2)}+v_{1,0}p^{(2,1)}+p^{(2,0)}+v_{0,-1}p^{(2,-1)}+v_{0,-2}p^{(2,-2)}=0.$$

The straightforward computation shows that if this symmetry exists then it must be of the form

$$\begin{aligned} V_t &= ff_1(V_2-v_{2,0}V)+f(\rho_1^{(1)}+\rho^{(1)})(V_1-v_{1,0}V)\\ &\quad +\kappa h(\tilde\rho_{-1}^{(1)}+\tilde\rho^{(1)}+\tilde\kappa)(V_{-1}-v_{0,-1}V)+\kappa hh_{-1}(V_{-2}-v_{0,-2}V) \end{aligned} \tag{12}$$

with some indeterminate integration constants $\kappa,\tilde\kappa$. Although the use of this explicit formula gives no essential advantage in solving the classification problem, it is useful as a final check of integrability of the obtained lattices.

3 Analysis of the integrability conditions

3.1 First step

Consider the first pair of integrability conditions (7), (8)

$$D_x(\log f)\in \mathrm{Im}(T-1),\quad D_x(\log h)\in \mathrm{Im}(T-1). \tag{13}$$

It is easy to obtain the following equations as a corollary:

$$\frac{f_{v_{1,-1}}}{f^2}+\frac{h}{f}T\left(\frac{f_{v_{1,-1}}}{f^2}\right)=0,\qquad \frac{h_{v_{1,-1}}}{h^2}+\frac{f}{h}T^{-1}\left(\frac{h_{v_{1,-1}}}{h^2}\right)=0. \tag{14}$$

Indeed, the terms containing scalar products $v_{k,k-3}$ appear only by differentiating $v_{1,-1}$ with respect to x:

$$\begin{aligned} D_x(\log f) &= \frac{f_{v_{1,-1}}}{f}D_x(v_{1,-1})+\dots=\frac{f_{v_{1,-1}}}{f}(f_1v_{2,-1}+h_{-1}v_{1,-2})+\dots\\ &\overset{\mathrm{Im}(T-1)}{\simeq}\left(\frac{f_{v_{1,-1}}}{f}f_1+T\left(\frac{f_{v_{1,-1}}}{f}\right)h\right)v_{2,-1}+\dots \end{aligned}$$

and the first equation (14) follows. This computation is actually equivalent to applying of *variational derivative* $\delta/\delta_{v_{3,0}}$ defined by formula

$$\frac{\delta a}{\delta v_{j,0}}=\frac{\partial}{\partial v_{j,0}}\sum_{k=-\infty}^{\infty}T^k(a),\quad j=1,2,\dots$$

The use of this notion makes the computations more algorithmic, due to the equality

$$\mathbb{C}\oplus \mathrm{Im}(T-1)=\bigcap_{j=1}^{\infty}\ker\frac{\delta}{\delta v_{j,0}}$$

which is proven along the same lines as in scalar case [2].

Statement 3. *The dependence of the coefficients of the lattice on $v_{1,-1}$ may be one of the following:*

$$\textit{Case 1.}\qquad f=\frac{a(v_{0,-1})}{v_{1,-1}+b(v_{1,0},v_{0,-1})},\qquad h=-\frac{a(v_{1,0})}{v_{1,-1}+b(v_{1,0},v_{0,-1})},$$
$$\textit{Case 2.}\qquad f=f(v_{1,0},v_{0,-1}),\quad h=h(v_{1,0},v_{0,-1}).$$

Proof. First equation (14) implies that $f_{v_{1,-1}}/f^2$ may depend on $v_{0,-1}$ only. If $f_{v_{1,-1}}\neq 0$ then we come to the Case 1. If $f_{v_{1,-1}}=0$ then $h_{v_{1,-1}}=0$ as well, in virtue of the second equation (14), and we come to the Case 2. $\square$

Conditions (13) are far from being exhausted by this statement. We will see that in the Case 1 they allow to define functions a, b as well.

3.2 Case 1: $f_{v_{1,-1}}\neq 0$

Notice that in this case the relation (11) at $j=0$ is satisfied with $\hat\sigma^{(0)}=-\log a(v_{0,-1})$. This means that conditions (13) are equivalent to each other and we may consider only the first one. Applying of $\delta/\delta_{v_{2,0}}$ to it is a rather tedious task. The resulting equation is polynomial in variables $v_{k+2,k}$ and vanishing of the coefficients brings to a certain overdetermined system for functions a and b. It is convenient to introduce the auxiliary functions

$$y(v)=\frac{1-v^2}{a^2(v)},\quad c(u,v)=\frac{b(u,v)+uv}{a(u)a(v)}\tag{15}$$

and to denote $u=v_{1,0}$, $v=v_{0,-1}$, $w=v_{-1,-2}$. This allows to rewrite the system in a relatively compact form as follows:

$$c(u,v)(a'(u)-a'(v))=(a(u)y(u))_u-(a(v)y(v))_v,\tag{16}$$
$$a(u)(c+y(u))c_u-a(v)(c+y(v))c_v=\frac{u(c-y(v))}{a(u)}-\frac{v(c-y(u))}{a(v)},\quad c=c(u,v),\tag{17}$$
$$(c(v,w)+y(v))(2c(u,v)+y(v))_v=(c(u,v)+y(v))(2c(v,w)+y(v))_v.\tag{18}$$

At first, we will prove that all solutions of equation (18) are:

$$\text{(i)}\qquad 2c(u,v)=2\alpha-y(u)-y(v),$$
$$\text{(ii)}\qquad c(u,v)=\alpha z(u)z(v)+\beta,\quad y(v)=\gamma z^2(v)-\beta,\quad z'\neq 0$$

where α,β,γ are arbitrary constants.

If $c(v,w)+y(v)=0$ or $c(u,v)+y(v)=0$ then (18) is reduced to the equation

$$0=(y(u)-y(v))y'(v),$$

hence $y(v)=-\beta$, $c(u,v)=\beta$, a special case of solution (ii).

If $(c(v,w)+y(v))(c(u,v)+y(v))\neq 0$ then the variables in (18) can be separated:

$$\frac{(2c(u,v)+y(u))_u}{c(u,v)+y(u)}=2k(u),\quad \frac{(2c(u,v)+y(v))_v}{c(u,v)+y(v)}=2k(v)\tag{19}$$

and as a corollary we obtain $c_{uv} = k(u)c_v = k(v)c_u$. The case $k = 0$ corresponds to the solution (i). At $k \neq 0$ we get $c = C(K(u) + K(v))$, $K' = k$, $C'' = C'$, whence $c = \alpha z(u)z(v) + \beta$, where $z' = kz$. Moreover, both equations (19) are reduced to the relation

$$y'(v) = \frac{2z'(v)}{z(v)}(y(v) + \beta)$$

and we get (ii) by integration. Now we consider both types of solutions separately and come to the following statement.

Statement 4. *The solutions $a = a(v)$, $b = b(u, v)$ of the system (15)–(18) are exhausted, up to the scaling $a \to \mathrm{const}\, a$, by the following list:*

$$a = v - 1/v, \quad b = -uv; \tag{20}$$

$$a^2 - kva + v^2 - 1 = 0, \quad b = a(u)a(v) - uv; \tag{21}$$

$$a = v + \varepsilon, \quad b = -1; \tag{22}$$

$$a = v + \varepsilon, \quad b = (u + \varepsilon)(v + \varepsilon)\left(\sqrt{\Big(\frac{u - \varepsilon}{u + \varepsilon} - k\Big)\Big(\frac{v - \varepsilon}{v + \varepsilon} - k\Big)} + k\right) - uv; \tag{23}$$

$$a = v + \varepsilon, \quad b = 1 + \varepsilon(u + v) + k\sqrt{(u + \varepsilon)(v + \varepsilon)} \tag{24}$$

where $\varepsilon = \pm 1$ and k is an arbitrary constant.

Proof. Solutions of type (i). Applying $\partial_u \partial_v$ to (16) yields

$$y'(u)a''(v) = y'(v)a''(u).$$

If $y' = 0$ then scaling allows to set $y = 1$, $a^2(v) = 1 - v^2$ and then (16) implies that $c = 1$. Equation (17) becomes identically true in virtue of these relations and we arrive to solution (21) at $k = 0$.

If $y' \neq 0$ then $a' = \mu y + \nu$. The variables in (16) are now separated and we obtain the overdetermined ODE system for the functions $a = a(v)$, $y = y(v)$:

$$ay' = R(y) = -\frac{3}{2}\mu y^2 + (\alpha\mu - \nu)y + \lambda, \quad a' = S(y) = \mu y + \nu, \quad ya^2 = 1 - v^2. \tag{25}$$

Differentiation yields

$$a(2yS + R) = -2v, \quad S(2yS + R) + (2S + 2y\dot{S} + \dot{R})R + 2 = 0.$$

The polynomial on y in the l.h.s. of the latter equation must vanish identically since $y' \neq 0$. This gives the relations $\mu = 0$, $\lambda\nu = -1$ and moreover, the scaling allows to set $\nu = 1$. Now, the system (25) is reduced to equations

$$ay' = -y - 1, \quad a = v + \varepsilon, \quad a^2 y = 1 - v^2.$$

It is easy to prove that they are consistent at $\varepsilon^2 = 1$, and an intermediate substitution into (17) proves that $\alpha = 0$. The resulting solution is (22).

Solutions of type (ii). Applying $\partial_u\partial_v$ to (16) yields

$$\alpha\left(a'(u)-a'(v)+\frac{z(u)a''(u)}{z'(u)}-\frac{z(v)a''(v)}{z'(v)}\right)=0. \tag{26}$$

If $\alpha=0$ then $c=\beta$ and variables in equation (17) are separated:

$$\frac{(\beta-y(u))a(u)}{u}=\frac{(\beta-y(v))a(v)}{v}=\delta.$$

This relation turn the equation (16) into identity as well. Taking (15) into account, we obtain the equation $\beta a^2-\delta va+v^2-1=0$ for $a(v)$. This brings, up to the scaling, to the solutions (20), (21).

If $\alpha\neq 0$ then we set $\alpha=1$ without loss of generality. Equation (26) implies $a'=\mu/q+\nu$, then the variables in (16) are separated and we obtain the overdetermined ODE system for the functions $a=a(v)$, $z=z(v)$:

$$a'=\frac{\mu}{z}+\nu,\quad ((\gamma z^2-\beta)a)'-\frac{\mu\beta}{z}+\mu z=\lambda,\quad (\gamma z^2-\beta)a^2=1-v^2. \tag{27}$$

Notice that $\gamma\neq 0$: otherwise $-2\mu\beta/z+\mu z-\beta\nu=\lambda$ and since $z'\neq 0$, hence $\mu=0$; but then the equations $a'=\nu$, $\beta a^2=v^2-1$ are inconsistent. Therefore, second equation (27) can be rewritten as follows:

$$z'=\frac{1}{2\gamma a}\left(-\gamma\nu z-\mu(\gamma+1)+\frac{\lambda+\beta\nu}{z}+\frac{2\mu\beta}{z^2}\right).$$

Now, differentiating of third equation (27) brings, as in the previous case, to a polynomial equation for z which must be satisfied identically. This gives equations for the parameters:

$$(\gamma-1)\beta\mu=0,\quad (3\gamma-1)(\lambda+\beta\nu)\mu=0,\quad (\gamma-3)\mu\nu=0,\quad 4\gamma(\lambda\nu+1)+(\gamma-1)^2\mu^2=0.$$

Moreover, substitution into (17) gives additionally the equations

$$(\gamma+1)(\gamma-3)\beta\mu=0,\quad (\gamma^2-1)(\lambda+\beta\nu)=0,\quad (\gamma^2-1)\mu=0.$$

The solutions of the whole system are:

$$(\mu^2=1,\beta=0,\lambda=0,\nu=0,\gamma=-1),$$
$$(\mu=0,\nu=-1/\lambda,\gamma^2=1),\quad (\mu=0,\nu=-1/\lambda,\beta=\lambda^2).$$

The first one is unsuitable since it leads to $z'=0$. For the other two we set $\nu=1$, $\lambda=-1$, $a=v+\varepsilon$ without loss of generality. It is easy to check that (27) are consistent at $\varepsilon^2=1$ and we come to solutions (23) and (24), respectively. □

It can be proved straightforwardly that conditions (7) at $j=0$ are fulfilled for each solution (20)–(24), that is there exist quantities $\sigma^{(0)}$, $\tilde\sigma^{(0)}$ which turn them into identities. It is sufficient to compute only $\sigma^{(0)}$, due to the relation $\tilde\sigma^{(0)}_{-1}=D_x(\hat\sigma^{(0)})-\sigma^{(0)}$ where $\hat\sigma^{(0)}=-\log a(v_{0,-1})$. Practically, this computation is based on the "summation by parts" algorithm, see e.g. [2, Theorem 1]. After finding $\sigma^{(0)}$ one can continue the integrability test with the next pair of densities (9). It turns out that in all cases except for (24) the second

integrability condition is fulfilled automatically. In the case (24) we obtain the restriction $k^3 - 4k = 0$ on the values of parameter. In more details, the density $\rho^{(1)}$ is in this case of the form

$$\rho^{(1)} = \frac{f_{-1}}{v_{-1,-2}+\varepsilon}(v_{0,-2}-1) + \frac{ff_{-1}}{v_{-1,-2}+\varepsilon}\Big(v_{1,-2} - v_{1,0} + v_{0,-1} - v_{-1,-2} \\ - \frac{1}{2}\left(k\sqrt{v_{1,0}+\varepsilon} + 2\varepsilon\sqrt{v_{0,-1}+\varepsilon}\right)\left(k\sqrt{v_{-1,-2}+\varepsilon} + 2\varepsilon\sqrt{v_{0,-1}+\varepsilon}\right)\Big)$$

and it can be proven that $\delta D_x(\rho^1)/\delta_{v_{2,0}}$ vanishes if and only if the above constraint holds.

The computation of $\sigma^{(1)}$ and further check of the integrability conditions require the considerable efforts. Fortunately, it is possible to avoid these calculations by checking that the explicit formula (12) provide the higher symmetry indeed. This turns out to be true for (20)–(23) and (24) at $k = 0, \pm 2$ (with constants $\kappa = -1$, $\tilde{\kappa} = 0$ in all cases) and we come, respectively, to the lattices (V_1)–(V_5) in the List 1 below.

3.3 Case 2: $f_{v_{1,-1}} = 0$

Computations here are easier, but also more lengthy, since in some subcases we have to check up to three integrability conditions (7). However, the result of this search is somewhat disappointing: it consists of one lattice (V_6). By this reason we give only schematic account of this case.

Applying $\delta/\delta_{v_{2,0}}$ to (13) yields the equations

$$\begin{aligned} &\frac{h}{f}\left(T\left(\frac{f_{v_{0,-1}}}{f}\right) + \frac{f_{v_{1,0}}}{f}\right) + \frac{f_{v_{0,-1}}}{f} + T^{-1}\left(\frac{f_{v_{1,0}}}{f}\right) = 0, \\ &\frac{h}{f}\left(T\left(\frac{h_{v_{0,-1}}}{h}\right) + \frac{h_{v_{1,0}}}{h}\right) + \frac{h_{v_{0,-1}}}{h} + T^{-1}\left(\frac{h_{v_{1,0}}}{h}\right) = 0. \end{aligned} \tag{28}$$

In turn, differentiating this with respect to $v_{2,1}$ yields

$$(\log f)_{v_{1,0},v_{0,-1}} = 0, \quad (\log h)_{v_{1,0},v_{0,-1}} = 0 \quad \Rightarrow \quad f = T(a)b, \quad h = T(c)d$$

where a, b, c, d are functions on $v_{0,-1}$. Now, the variables in equations (28) are separated and we come to relations

$$\frac{(ab)'}{ab}\cdot\frac{c}{a} = \mu, \quad \frac{(ab)'}{ab}\cdot\frac{b}{d} = -\mu, \quad \frac{(cd)'}{cd}\cdot\frac{b}{d} = \nu, \quad \frac{(cd)'}{cd}\cdot\frac{c}{a} = -\nu$$

with some constants μ, ν. If $ab + cd \neq 0$ then $(ab)' = (cd)' = 0$, so that two cases are possible, up to the scaling:

$$\begin{aligned} &\text{(i)} \qquad b = p/a, \quad c = ap/p', \quad d = -p'/a, \quad p' \neq 0; \\ &\text{(ii)} \qquad a = \alpha/b, \quad d = 1/c. \end{aligned}$$

In the case (i), applying $\delta/\delta_{v_{1,0}}$ to (13) brings to certain overdetermined system for functions a, p. It is convenient to analyze this system taking into account some additional information (namely, the equation $pp'' = \mathrm{const}(p')^2$) which can be obtained either from the integrability

$$V_x=\frac{a(V_1-v_{1,0}V)+a_1(v_{0,-1}V-V_{-1})}{v_{1,-1}-v_{1,0}v_{0,-1}}\,,\quad a=v_{0,-1}-\frac{1}{v_{0,-1}}; \tag{V_1}$$

$$V_x=\frac{a(V_1-v_{1,0}V)+a_1(v_{0,-1}V-V_{-1})}{v_{1,-1}-v_{1,0}v_{0,-1}+aa_1}\,,\quad a^2-2kv_{0,-1}a+v_{0,-1}^2-1=0; \tag{V_2}$$

$$V_x=\frac{(v_{0,-1}+\varepsilon)(V_1+\varepsilon V)-(v_{1,0}+\varepsilon)(V_{-1}+\varepsilon V)}{v_{1,-1}-1}\,; \tag{V_3}$$

$$V_x=\frac{(v_{0,-1}+\varepsilon)(V_1+\varepsilon V)-(v_{1,0}+\varepsilon)(V_{-1}+\varepsilon V)}{v_{1,-1}-v_{1,0}v_{0,-1}+(v_{1,0}+\varepsilon)(v_{0,-1}+\varepsilon)(k+pp_1)}\,,\quad p=\sqrt{\frac{v_{0,-1}-\varepsilon}{v_{0,-1}+\varepsilon}-k}; \tag{V_4}$$

$$V_x=\frac{(v_{0,-1}+\varepsilon)(V_1+\varepsilon V)-(v_{1,0}+\varepsilon)(V_{-1}+\varepsilon V)}{v_{1,-1}+\varepsilon(v_{1,0}+v_{0,-1})+1+k\sqrt{v_{1,0}+\varepsilon}\sqrt{v_{0,-1}+\varepsilon}}\,,\quad k=0,\pm 2; \tag{V_5}$$

$$V_x=\frac{V_1+\delta V}{v_{1,0}+\delta}-\frac{V_{-1}+\delta V}{v_{0,-1}+\delta}\,,\quad \delta=0,\pm 1. \tag{V_6}$$

List 1: Integrable lattices, $\langle V,V\rangle=1$, $v_{m,n}=\langle V_m,V_n\rangle$, $\varepsilon=\pm 1$.

condition (11) at $j=1$ or from the next pair of conservation laws (7), (9). This allows to prove that functions $a(v),p(v)$ may be the following:

$$a=p=\frac{1}{v+\delta};\qquad a=1,\ p=v+\delta;\qquad a=v,\ p=v^3.$$

The check of conservation laws (7), (9) for the first solution proves that δ must take the values $\pm 1,0$ and leads to the lattice (V_6), while two other solutions do not pass the test.

In the case (ii) the first pair of integrability conditions (7), (8) is fulfilled for any α,b,c. The further analysis proves that conditions (7), (9) are fulfilled if $\alpha=1$ and either $b(v)=c(v)=\sqrt{v+\delta}$ or $b=c=1$. However, the next conditions (7), (10) fail in both cases, so that this case turns out to be empty.

4 The list of integrable lattices

Theorem 1. *If isotropic Volterra type lattice on the sphere $\langle V,V\rangle=1$ satisfies integrability conditions (7)–(10) then it coincides with one of the lattices from the List 1, up to scaling of x. Each lattice from this list possesses at least one higher symmetry of the form (12).*

Remark 2. The lattices corresponding to the different signs of ε or δ are equivalent modulo flip map $V_n\to(-1)^nV_n$. The lattice (V_2) at $k=\pm 1$ coincides with (V_5) at $k=0$.

The lattice (V_6) is the discrete Heisenberg spin chain introduced in [15], see also [16, 17] where the applications to the discrete geometry were considered and [8] where the anisotropic version (see Section 7) was studied. It can be written (at $\delta=1$ and after scaling x) as

$$V_x=\frac{V_1+V}{|V_1+V|^2}-\frac{V+V_{-1}}{|V+V_{-1}|^2}\,. \tag{29}$$

In this form, the constraint $\langle V,V\rangle=1$ is not necessary for integrability. This lattice and its higher symmetry (12) can be written compactly as

$$V_x=(T-1)(W),\quad V_t=(T-1)P_W(W_1-W_{-1}),\quad W=(V+V_{-1})^{-1}$$

by use of the operations

$$A^{-1}=\frac{1}{\langle A,A\rangle}A,\qquad P_A(B)=2\langle A,B\rangle A-\langle A,A\rangle B.$$

The variable U satisfies the polynomial lattices

$$W_x=-P_W(W_1-W_{-1}),\quad W_t=-P_W(P_{W_1}(W_2+W)-P_{W_{-1}}(W+W_{-2}))$$

which are integrable not only in the vector case, but also in more general setting related to Jordan triple systems [18].

The lattices (V_1)–(V_5) are new, up to the author's knowledge. The lattice (V_3) is related to (V_6) by composition of difference substitution and reduction. Namely, first we can resolve the constraint $\langle V,V\rangle=1$ by use of stereographic projection as explained in Introduction. This brings (V_3) at $\varepsilon=-1$ to the form

$$U_x=\frac{|U-U_{-1}|^2(U_1-U)+|U_1-U|^2(U-U_{-1})}{|U_1-U_{-1}|^2}$$

and then substitution $\tilde V=U-U_{-1}$ brings it to the lattice

$$\tilde V_x=\frac{|\tilde V|^2\tilde V_1+|\tilde V_1|^2\tilde V}{|\tilde V_1+\tilde V|^2}-\frac{|\tilde V_{-1}|^2\tilde V+|\tilde V|^2\tilde V_{-1}}{|\tilde V+\tilde V_{-1}|^2}.$$

This is not the same lattice as (29), however it is obvious that both lattices admit the reduction on sphere which brings them to the lattice (V_6). The question on the substitutions for the other lattices from the list is so far open.

5 Associated partial differential equations

The very general observation due to Levi [19] is that a higher symmetry of an integrable lattice gives rise to some PDE after elimination of the discrete variable n. The lattice itself is now interpreted as Bäcklund transformation for this PDE. The examples of such relation can be found in [11, 12] and many other works. In particular, the integrable Volterra lattices (4) are associated with some systems of nonlinear Schrödinger type. There are known also many results on the multifield analogs of NLS-type systems, see e.g. [20, 21, 22, 23], however their classification is far from being completed. The list of vector Volterra lattices provides several new examples of such systems.

The elimination of the discrete variable is done as follows. The equations (2), (3) imply the corollaries

$$\langle V_x,V_1\rangle=(1-v^2_{1,0})f+(v_{1,-1}-v_{1,0}v_{0,-1})h,$$
$$\langle V_x,V_x\rangle=(1-v^2_{1,0})f^2+2(v_{1,-1}-v_{1,0}v_{0,-1})fh+(1-v^2_{0,-1})h^2.$$

We assume that these equations can be solved with respect to the scalar products $v_{1,-1}$, $v_{0,-1}$ (this is true for all lattices from the List 1). Then equation (2) can be rewritten in the form

$$V_{-1} = \tilde{f}V_1 + \tilde{g}V + \tilde{h}V_x \tag{30}$$

with coefficients depending on the scalar products of vectors V_1, V, V_x. Analogously,

$$V_2 = \hat{f}V_{1,x} + \hat{g}V_1 + \hat{h}V.$$

Iteration of these formulae allows to express all vectors V_n through the vectors $U = V_1$, V and their derivatives. As a result, the symmetry (12) gives rise to a system of the form

$$\begin{cases} U_t = U_{xx} + \alpha U_x + \beta V_x + \gamma U + \delta V, \\ -V_t = V_{xx} + \tilde{\alpha}U_x + \tilde{\beta}V_x + \tilde{\gamma}U + \tilde{\delta}V, \end{cases} \qquad \langle U, U\rangle = \langle V, V\rangle = 1 \tag{31}$$

with coefficients depending on the scalar products of U, U_x, V and V_x. The equation (30) becomes an explicit Bäcklund auto-transformation

$$U_{-1} = V, \quad V_{-1} = \tilde{f}U + \tilde{g}V + \tilde{h}V_x$$

of this system. Converse is not true: not any integrable system (31) admits auto-BT of such form. Classification problem for this type of equations may be difficult, since even the simplest lattices from our list correspond to rather cumbersome systems (31). Few instances are given below. In the case (V_6) at $\delta = \pm 1$ we come to the system

$$\begin{aligned} U_t &= U_{xx} - \frac{2\langle U_x, V\rangle + 4\delta}{\langle U, V\rangle + \delta}U_x + \frac{2V_x}{\langle U, V\rangle + \delta} + \left(\frac{\langle U_x, U_x\rangle}{\langle U, V\rangle + \delta} - \frac{2\langle U, V_x\rangle}{(\langle U, V\rangle + \delta)^2}\right)(\delta U + V), \\ -V_t &= V_{xx} - \frac{2\langle U, V_x\rangle - 4\delta}{\langle U, V\rangle + \delta}V_x - \frac{2U_x}{\langle U, V\rangle + \delta} + \left(\frac{\langle V_x, V_x\rangle}{\langle U, V\rangle + \delta} + \frac{2\langle U_x, V\rangle}{(\langle U, V\rangle + \delta)^2}\right)(U + \delta V), \end{aligned}$$

while (V_6) at $\delta = 0$ corresponds to the system

$$\begin{aligned} U_t &= U_{xx} - \frac{2\langle U_x, V\rangle\langle U, V\rangle + 2}{\langle U, V\rangle^2}U_x + \left(\langle U_x, U_x\rangle + \frac{2\langle U_x, V\rangle}{\langle U, V\rangle}\right)U + \left(\frac{2V}{\langle U, V\rangle}\right)_x, \\ -V_t &= V_{xx} - \frac{2\langle U, V_x\rangle\langle U, V\rangle - 2}{\langle U, V\rangle^2}V_x + \left(\langle V_x, V_x\rangle - \frac{2\langle U, V_x\rangle}{\langle U, V\rangle}\right)V - \left(\frac{2U}{\langle U, V\rangle}\right)_x. \end{aligned}$$

The lattice (V_3) is associated with the system

$$\begin{aligned} U_t = U_{xx} &- 2\left(\frac{\langle U, V_x\rangle\langle U_x, V\rangle}{(\langle U, V\rangle + \varepsilon)^2} - \frac{\langle U_x, V_x - V\rangle}{\langle U, V\rangle + \varepsilon}\right)U_x - \frac{\langle U_x, U_x\rangle}{\langle U, V\rangle + \varepsilon}V_x \\ &+ \frac{\langle U_x, U_x\rangle}{\langle U, V\rangle + \varepsilon}\left(1 + \frac{\langle U, V_x\rangle}{\langle U, V\rangle + \varepsilon}\right)(\varepsilon U + V), \\ -V_t = V_{xx} &+ 2\left(\frac{\langle U, V_x\rangle\langle U_x, V\rangle}{(\langle U, V\rangle + \varepsilon)^2} - \frac{\langle V_x, U_x + U\rangle}{\langle U, V\rangle + \varepsilon}\right)V_x + \frac{\langle V_x, V_x\rangle}{\langle U, V\rangle + \varepsilon}U_x \\ &+ \frac{\langle V_x, V_x\rangle}{\langle U, V\rangle + \varepsilon}\left(1 - \frac{\langle U_x, V\rangle}{\langle U, V\rangle + \varepsilon}\right)(U + \varepsilon V). \end{aligned}$$

6 Presymplectic structure

The bi-Hamiltonian structure of the scalar Volterra lattice is well known, see e.g. [24]. In the vector case the question is more difficult and it requires further investigation. However, the following statement shows that all lattices under scrutiny possess at least some uniform presymplectic structure.

Statement 5. *Any lattice (V*$_1$*)–(V*$_6$*) can be written in presymplectic form*

$$SV_x=\frac{\delta H}{\delta V}+\lambda V,\quad H=\rho^{(0)}=\log f(v_{1,-1},v_{1,0},v_{0,-1}) \tag{32}$$

where S is a certain skew-symmetric operator of the form

$$S=pT^{-1}-p_1T-qV_{-1}V^\top T^{-1}+q_1V_1V^\top T+r(V_1V_{-1}^\top-V_{-1}V_1^\top), \tag{33}$$

λ is Lagrange multiplier corresponding to the constraint $\langle V,V\rangle=1$ and operator $UV^\top$ acts accordingly to the formula $UV^\top(W)=U\langle V,W\rangle$.

Proof. The equation (32) is equivalent to

$$\begin{gathered}(pT^{-1}-p_1T)(fV_1+gV+hV_{-1})-V_{-1}(qT^{-1}+r)(f+v_{1,0}g+v_{1,-1}h)\\ +V_1(r+q_1T)(v_{1,-1}f+v_{0,-1}g+h)-\lambda V\\ =T\left(\frac{f_{v_{1,-1}}}{f}V_1+\frac{f_{v_{0,-1}}}{f}V\right)+\frac{f_{v_{1,0}}}{f}V_1+\frac{f_{v_{0,-1}}}{f}V_{-1}+T^{-1}\left(\frac{f_{v_{1,0}}}{f}V+\frac{f_{v_{1,-1}}}{f}V_{-1}\right).\end{gathered}$$

Equating the coefficients at $V,V_{\pm2}$ yields

$$\lambda=pf_{-1}-p_1h_1,\quad p=-f_{v_{1,-1}}/f^2,\quad pf+p_1h=0.$$

The first two equations are just definitions of λ and p while the latter one is fulfilled for the lattices from the list in virtue of (14). Equations for the rest coefficients give the system for q and r of the form

$$Ar+A_1q_1=C,\quad Br+B_{-1}q=D \tag{34}$$

where

$$\begin{gathered}A=v_{1,-1}f+v_{0,-1}g+h,\quad B=f+v_{1,0}g+v_{1,-1}h,\\ C=p_1g_1+(\log f_1f)_{v_{1,0}},\quad D=pg_{-1}-(\log ff_{-1})_{v_{0,-1}}.\end{gathered}$$

Elimination of one of the unknown functions, say r, brings (34) to the form

$$(T-1)(AB_{-1}q)=BC-AD.$$

This means that the system (34) is solvable if and only if $BC-AD\in\operatorname{Im}(T-1)$. Remarkably, this condition is equivalent exactly to $D_x(\log f)\in\operatorname{Im}(T-1)$, as an easy check proves, and therefore it is true for all lattices from the List 1. □

The concrete expressions for the coefficients q,r may be rather cumbersome (it is clear from the proof that they are related somehow with the quantity $\sigma^{(0)}$). The answer is very simple for the lattice (V_3):

$$p=\frac{1}{v_{0,-1}+\varepsilon},\quad q=\frac{1}{(v_{0,-1}+\varepsilon)^2},\quad r=0. \tag{35}$$

The formula $\langle U, SW\rangle = \Omega(U,W)$ relates operator S with 2-form

$$\Omega = \sum_n \big(p_n\langle dV_n \overset{\wedge}{,} dV_{n-1}\rangle + q_n\langle V_n, dV_{n-1}\rangle \wedge \langle V_{n-1}, dV_n\rangle \\ +r_n\langle V_{n+1}, dV_n\rangle \wedge \langle V_{n-1}, dV_n\rangle\big)$$

where $\langle \alpha \overset{\wedge}{,} \beta\rangle(U,W) := \langle\alpha(U),\beta(W)\rangle - \langle\alpha(W),\beta(U)\rangle$. It is easy to see that this form is exact in the case (35), namely $\Omega = d\sum_n p_n\langle V_n, dV_{n-1}\rangle$. Therefore $d\Omega = 0$, that is operator S is symplectic indeed. Unfortunately, this is not true in the general case.

It is also worth to notice that the representation (32) can be replaced with a linear pencil by assuming that Hamiltonian is of the form $H = \rho^{(0)} + \kappa\rho$, where ρ is some additional conserved density depending on $v_{1,0}$ (it does not belong to the sequence (7), however it turns out that such densities exist for all lattices under consideration). Operator S also acquires linear dependence on κ, preserving the same structure (33). We bring the explicit formulae only for the relatively simple case of lattice (V_1):

$$\rho^{(0)} = \log\frac{a}{v_{1,-1} - v_{1,0}v_{0,-1}}, \quad \rho = \log v_{1,0}, \quad p = \frac{1}{a}, \quad a = v_{0,-1} - \frac{1}{v_{0,-1}},$$

$$q = \frac{1}{a^2} + (\kappa-1)\frac{(v_{1,-1} - v_{1,0}v_{0,-1})(v_{0,-2} - v_{0,-1}v_{-1,-2})}{av_{0,-1}\left(v_{1,-1} - \dfrac{v_{0,-1}}{v_{1,0}}\right)\left(v_{0,-2} - \dfrac{v_{0,-1}}{v_{-1,-2}}\right)},$$

$$r = \frac{1}{a_1 a} + (\kappa-1)\frac{v_{1,-1} - v_{1,0}v_{0,-1}}{\left(v_{1,-1} - \dfrac{v_{0,-1}}{v_{1,0}}\right)\left(v_{1,-1} - \dfrac{v_{1,0}}{v_{0,-1}}\right)}.$$

Operator S is not symplectic here. We see also that its simplest form corresponds to the Hamiltonian $\rho^{(0)} + \rho$ rather than $\rho^{(0)}$, but this may be not so for the other lattices.

7 Concluding remarks

The goal of the present paper was to solve some classification problem; such important things as difference substitutions, Lax pairs, Bäcklund transformations, explicit solutions and so on have not been considered. These open problems require, probably, more individual investigation for each member of the obtained list. From the author's point of view, the question on the Hamiltonian properties of the vectorial equations is among the most intriguing ones.

It was mentioned in Introduction that the assumption (iii) can be removed by use of stereographic projection. Another interesting setting is related with the variables on the cone $\langle V, V\rangle = 0$ instead of the sphere. At first sight, this constraint may be treated as a limiting case, but actually it defines some independent class of equations. In particular, in this case the coefficient g is not expressed through f, h and we also have no explicit formula like (12) for the symmetry. An interesting example here is the lattice

$$V_x = \frac{1}{v_{1,-1}}(v_{0,-1}V_1 - v_{1,0}V_{-1}) + b(v_{1,-1}, v_{1,0}, v_{0,-1})V, \quad v_{n,n} = 0.$$

It is likely that it satisfies the infinite sequence of integrability conditions (7) at arbitrary b, but (local) symmetries exist only if $b_{v_{1,-1}} = 0$.

The other possible generalizations are related with the condition (i). The simplest anisotropic lattice is analog of (V_6)

$$V_x = \langle V, KV\rangle \left(\frac{V_1+V}{1+\langle V_1,V\rangle} - \frac{V+V_{-1}}{1+\langle V,V_{-1}\rangle}\right), \quad \langle V,V\rangle = 1$$

where K is an arbitrary symmetric operator. This lattice is closely related to many other integrable equations, among them Sklyanin lattice and Landau-Lifshitz equation [8]. The classification problem in the anisotropic case can be in principle solved along the same lines (cf [3, 4] in the continuous case), however technically it is much more difficult since coefficients acquire dependence on the additional variables $\tilde{v}_{m,n} = \langle V_m, KV_n\rangle$. It is interesting to consider also the asymmetric scalar product ($v_{m,n} \neq v_{n,m}$), however the examples of this type are not known at the moment.

Acknowledgements. This work was supported by the Russian Foundation for Basic Researches under grant # 06-01-92051-KE-a.

References

[1] R.I. Yamilov. Classification of discrete evolution equations. *Usp. Mat. Nauk* ***38:6*** (1983) 155–156.

[2] R.I. Yamilov. Symmetries as integrability criteria for differential-difference equations. *J. Phys. A* ***39*** (2006) R541–623.

[3] A.G. Meshkov, V.V. Sokolov. Integrable evolution equations on the N-dimensional sphere. *Comm. Math. Phys.* ***232:1*** (2002) 1–18.

[4] A.G. Meshkov, V.V. Sokolov. Classification of integrable divergent N-component evolution systems. *Theor. Math. Phys.* ***139:2*** (2004) 609–622.

[5] V.V. Sokolov, T. Wolf. Classification of integrable polynomial vector evolution equations. *J. Phys. A* ***34*** (2001) 11139–11148.

[6] T. Tsuchida, T. Wolf. Classification of polynomial integrable systems of mixed scalar and vector evolution equations. I. *J. Phys. A* ***38*** (2005) 7691–7733.

[7] S.C. Anco, T. Wolf. Some symmetry classifications of hyperbolic vector evolution equations. *J. Nonl. Math. Phys.* ***12:1*** (2005) 1–19.

[8] V.E. Adler. Discretizations of the Landau-Lifshitz equation. *Theor. Math. Phys.* ***124:1*** (2000) 897–908.

[9] V.V. Sokolov, A.B. Shabat. Classification of integrable evolution equations. *Sov. Sci. Rev. C / Math. Phys. Rev.* ***4*** (1984) 221–280.

[10] A.V. Mikhailov, A.B. Shabat, R.I. Yamilov. The symmetry approach to classification of nonlinear equations. Complete lists of integrable systems. *Russ. Math. Surveys* ***42:4*** (1987) 1–63.

[11] A.B. Shabat, R.I. Yamilov. Lattice representations of integrable systems. *Phys. Lett. A* ***130:4,5*** (1988) 271–275.

[12] A.B. Shabat, R.I. Yamilov. Symmetries of nonlinear chains. *Len. Math. J.* ***2:2*** (1991) 377–399.

[13] D. Levi, R.I. Yamilov. Conditions for the existence of higher symmetries of evolutionary equations on the lattice. *J. Math. Phys.* ***38*** (1997) 6648–6674.

[14] V.E. Adler, A.B. Shabat, R.I. Yamilov. Symmetry approach to the integrability problem. *Theor. Math. Phys.* ***125:3*** (2000) 1603–1661.

[15] O. Ragnisco, P.M. Santini. A unified algebraic approach to integral and discrete evolution equations. *Inverse Problems* ***6*** (1990) 441–452.

[16] A.I. Bobenko. Discrete integrable systems and geometry. *In: XIIth Int. Congress of Math. Phys. (ICMP'97, Brisbane), 219–226*, Cambridge: Internat. Press, 1999.

[17] A.I. Bobenko, Yu.B. Suris. Discrete time Lagrangian mechanics on Lie groups, with an application to the Lagrange top. *Comm. Math. Phys.* ***204*** (1999) 147–188.

[18] V.E. Adler, S.I. Svinolupov, R.I. Yamilov. Multi-component Volterra and Toda type integrable equations. *Phys. Lett. A* ***254*** (1999) 24–36.

[19] D. Levi. Nonlinear differential difference equations as Bäcklund transformations. *J. Phys. A* ***14:5*** (1981) 1083–1098.

[20] S.I. Svinolupov. Generalized Schrödinger equations and Jordan pairs. *Comm. Math. Phys.* ***143*** (1992) 559–575.

[21] V.V. Sokolov, S.I. Svinolupov. Vector-matrix generalizations of classical integrable equations. *Theor. Math. Phys.* ***100*** (1994) 959–962.

[22] S.I. Svinolupov, V.V. Sokolov. Deformations of Jordan triple systems and integrable equations. *Theor. Math. Phys.* ***108:3*** (1996) 388–392.

[23] I.T. Habibullin, V.V. Sokolov, R.I. Yamilov. Multi-component integrable systems and nonassociative structures, pp. 139–168 in: *Nonlinear Physics: Theory and Experiment, Lecce'95* (E. Alfinito, M. Boiti, L. Martina, F. Pempinelli eds). Singapore: World Scientific, 1996.

[24] L.A. Takhtajan, L.D. Faddeev. Hamiltonian approach in the soliton theory. Moscow: Nauka, 1986.